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The absence of normalizable time-periodic solutions for the Dirac equation in the Kerr–Newman–dS black hole background

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Abstract
We consider the Dirac equation on the background of a Kerr–Newman–de Sitter black hole. By performing variable separation, we show that no time-periodic and normalizable solution of the Dirac equation is allowed, which amounts to the absence of quantum bound states for the Dirac Hamiltonian. This conclusion holds true even for extremal black holes. With respect to previously considered cases, the novelty is represented by the presence, in addition to a black hole event horizon, of a cosmological (non-degenerate) event horizon, which is at the root of the possibility to draw a conclusion on the aforementioned topic in a straightforward way even in the extremal case.

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1. Introduction

In this paper, we extend results obtained for the Dirac equation on the background of a Kerr–Newman–AdS black hole [1] to the case of a Kerr–Newman–de Sitter black hole. The main differences between the AdS and the dS case are the presence of a positive cosmological constant in the dS case (to be compared with the negative cosmological constant of the AdS case), the replacement of a boundary-like behavior of infinity in the AdS case with the presence of a further (non-degenerate) event horizon in the dS case: the cosmological horizon appears. In the dS case, problems with the lack of global hyperbolicity disappear and a good behavior of the wave operators is shown to be allowed. From the point of view of quantum field theory on the given background, with respect to the case of a single event horizon, further difficulties arise, due to the presence in the non-extremal case of two different background temperatures which make a rigorous analysis more difficult. We do not deal with this problem herein,

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and we limit ourselves to studying the problem of the absence of time-periodic normalizable solutions of the Dirac equation (which amounts to the absence of a point spectrum for the Dirac Hamiltonian, i.e. of quantum bound states [1]). The latter topic has given rise to a number of studies in the recent literature [2–10], mostly involved in black holes of the Kerr–Newman family, or still in the absence of a cosmological constant. We also considered this problem in the case of Kerr–Newman–AdS black holes [1]. In the aforementioned studies the absence of time-periodic normalizable solutions of the Dirac equation has been proved mainly in the non-extremal case. The extremal one has been shown to require further investigation, and in the Kerr–Newman case the existence of normalizable time-periodic solutions was proved in [5, 6].

It is a peculiar property of the background considered herein to forbid the existence of time-periodic normalizable solutions for the Dirac equation even in the extremal case, and this can be proved in a rather straightforward way. Naively, the presence of a cosmological event horizon, which is surely non-degenerate in our setting, does not allow normalizability of the solutions to be obtained near the cosmological horizon. This peculiar presence is shown to be at the root of the fact that the reduced radial Hamiltonian, obtained by variable separation, has an absolutely continuous spectrum which coincides with $\mathbb{R}$, and then no quantum bound state is allowed.

2. The Kerr–Newman–dS solution

The background geometry underlying our problem arises as follows. One first solves the Einstein–Maxwell equations with a cosmological constant, and next adds a Dirac field minimally coupled to the electromagnetic field. The Einstein–Maxwell action is

$$S[g_{\mu\nu}, A_\rho] = -\frac{1}{16\pi}\int (R-2\Lambda)\sqrt{-\det g}\,\mathrm{d}^4x - \frac{1}{16\pi}\int \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\sqrt{-\det g}\,\mathrm{d}^4x, \tag{2.1}$$

where $\Lambda = \frac{3}{l^2}$ is the positive cosmological constant, R is the scalar curvature and $F_{\mu\nu}$ is the field strength associated with the potential 1-form A:

$$F = \mathrm{d}A, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.2}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \qquad R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu}, \tag{2.3}$$

$$\Gamma^\mu_{\nu\rho} = \tfrac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}). \tag{2.4}$$

The equations of motion are

$$R_{\mu\nu} - \tfrac{1}{2}(R-2\Lambda)g_{\mu\nu} = -2\left(F^\rho_\mu F_{\nu\rho} - \tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{2.5}$$

$$\partial_\mu(\sqrt{-\det g}\,F^{\mu\nu}) = 0. \tag{2.6}$$

With respect to a set of vierbein 1-forms

$$\mathrm{e}^i = e^{\mathrm{i}}_\mu\,\mathrm{d}x^\mu, \qquad i = 0, 1, 2, 3, \tag{2.7}$$

we have

$$\mathrm{d}s^2 = g = \eta_{ij}\,\mathrm{e}^i \otimes \mathrm{e}^j, \qquad g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu, \tag{2.8}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the usual flat Minkowski metric, so that, as usual, we define the $so(1,3)$ valued spin connection 1-forms ω^i_j such that

$$\mathrm{d}\mathrm{e}^i + \omega^i_j \wedge \mathrm{e}^j = 0. \tag{2.9}$$

We will consider the following background solution.

The metric is (cf e.g. [11–13])

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left[dt - \frac{a\sin^2\theta}{\Xi}\,d\phi\right]^2 + \frac{\rho^2}{\Delta_r}\,dr^2 + \frac{\rho^2}{\Delta_\theta}\,d\theta^2 + \Delta_\theta\frac{\sin^2\theta}{\rho^2}\left[a\,dt - \frac{r^2+a^2}{\Xi}\,d\phi\right]^2, \tag{2.10}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Xi = 1 + \frac{a^2}{l^2}, \qquad z^2 = q_e^2 + q_m^2, \tag{2.11}$$

$$\Delta_\theta = 1 + \frac{a^2}{l^2}\cos^2\theta, \qquad \Delta_r = (r^2+a^2)\left(1 - \frac{r^2}{l^2}\right) - 2mr + z^2, \tag{2.12}$$

and the electromagnetic potential and field strength are

$$A = -\frac{q_e r}{\rho\sqrt{\Delta_r}}e^0 - \frac{q_m\cos\theta}{\rho\sqrt{\Delta_\theta}\sin\theta}e^1, \tag{2.13}$$

$$\begin{aligned} F = &-\frac{1}{\rho^4}[q_e(r^2 - a^2\cos^2\theta) + 2q_m ra\cos\theta]e^0\wedge e^2 \\ &+ \frac{1}{\rho^4}[q_m(r^2 - a^2\cos^2\theta) - 2q_e ra\cos\theta]e^3\wedge e^1, \end{aligned} \tag{2.14}$$

where we introduced the vierbein

$$e^0 = \frac{\sqrt{\Delta_r}}{\rho}\left(dt - \frac{a\sin^2\theta}{\Xi}\,d\phi\right), \tag{2.15}$$

$$e^1 = \frac{\sqrt{\Delta_\theta}\sin\theta}{\rho}\left(a\,dt - \frac{r^2+a^2}{\Xi}\,d\phi\right), \tag{2.16}$$

$$e^2 = \frac{\rho}{\sqrt{\Delta_r}}\,dr, \tag{2.17}$$

$$e^3 = \frac{\rho}{\sqrt{\Delta_\theta}}\,d\theta. \tag{2.18}$$

We are interested in the case where three real positive zeroes of Δ_r appear: a cosmological event horizon radius r_c, a black hole event horizon $r_+ < r_c$, a Cauchy horizon $r_- \leqslant r_+$, with the extremal case which is implemented when $r_- = r_+$ and the non-extremal case implemented otherwise. The following reparameterization of Δ_r is useful:

$$\Delta_r = \frac{1}{l^2}(r_c - r)(r - r_+)(r - r_-)(r + r_c + r_+ + r_-), \tag{2.19}$$

where the parameters m, z^2, a^2, l are replaced by r_c, r_+, r_-, l. One easily finds

$$\begin{aligned} m &= \frac{1}{2l^2}(r_c + r_+)(r_c + r_-)(r_+ + r_-), \\ a^2 &= l^2 - \left(r_c^2 + r_+^2 + r_-^2 + r_c r_+ + r_c r_- + r_+ r_-\right), \\ z^2 &= \frac{1}{l^2}r_c r_+ r_-(r_c + r_+ + r_-) - a^2. \end{aligned}$$

We note that the above reparameterization implies $a^2 < l^2$. As to the determinant $\mathcal{J}$ of the Jacobian matrix, we find

$$\mathcal{J} = -\frac{1}{2l^4}(r_c - r_+)(r_c - r_-)(r_+ - r_-)(2r_c + r_+ + r_-)(r_c + 2r_+ + r_-)(r_c + r_+ + 2r_-), \tag{2.20}$$

which is negative everywhere in the non-extremal case. In the extremal one, an analogous reparameterization exists, with the only caveat that the number of independent parameters is three (e.g. z^2, a^2, l).

As to the existence of black hole solutions for given values of the geometrical parameters, a study of the existence of zeroes for Δ_r is required (see also [12, 13]). A first observation is that, in order that there exist four real zeroes, it is necessary that $\frac{d^2\Delta_r}{dr^2}$ admits two real zeroes, and this leads again to the condition $l^2 - a^2 > 0$. Qualitatively, one can point out that for $m = 0$ the function Δ_r admits only two real zeroes $r_0 < r_c$ and two (symmetric) positive maxima and one positive minimum between them; for increasing $m > 0$, the minimum eventually intersects the r-axis, say at $m = m^-_{\text{crit}}$, providing the existence of two further zeroes $r_- \leqslant r_+$ which coincide for $m = m^-_{\text{crit}}$ (extremal black hole); an upper bound m^+_{crit} to m has still to be set, because the maximum on the right of the minimum eventually reaches the r-axis, where $r_+ = r_c$, and then for $m > m^+_{\text{crit}}$ again two real solutions remain. The aforementioned two critical situations are obtained by solving the system

$$\Delta_r = 0, \tag{2.21}$$

$$\Delta'_r = 0, \tag{2.22}$$

where $\Delta'_r := \frac{d\Delta_r}{dr}$, i.e. the equivalent system

$$\Delta_r - r\Delta'_r = 0, \tag{2.23}$$

$$\Delta'_r = 0. \tag{2.24}$$

Equation (2.23) amounts to

$$3r^4 - r^2(l^2 - a^2) + l^2(a^2 + z^2) = 0, \tag{2.25}$$

its solutions are

$$R_\pm = \sqrt{\frac{l^2 - a^2}{6} \pm \frac{1}{6}\sqrt{(l^2 - a^2)^2 - 12l^2(a^2 + z^2)}}, \tag{2.26}$$

and their existence, with $R_+ > R_-$, requires the condition $(l^2 - a^2)^2 - 12l^2(a^2 + z^2) > 0$, i.e.

$$\frac{a^2}{l^2} \leqslant 7 - 4\sqrt{3}, \tag{2.27}$$

which is sensibly more restrictive than $\frac{a^2}{l^2} < 1$. Then, from (2.24), one finds the corresponding critical values of the mass:

$$m^\pm_{\text{crit}} = \frac{R^2_\pm}{l^2}\left[l^2 - a^2 - 2R^2_\pm\right]. \tag{2.28}$$

Then we find the condition (together with (2.27)) to be satisfied

$$m^-_{\text{crit}} \leqslant m < m^+_{\text{crit}}, \tag{2.29}$$

with

$$m^\pm_{\text{crit}} = \frac{l}{3\sqrt{6}}\left(\left(1 - \frac{a^2}{l^2}\right) \pm \sqrt{\left(1 - \frac{a^2}{l^2}\right)^2 - \frac{12}{l^2}(a^2 + z^2)}\right)^{\frac{1}{2}} \times \left(2\left(1 - \frac{a^2}{l^2}\right) \mp \sqrt{\left(1 - \frac{a^2}{l^2}\right)^2 - \frac{12}{l^2}(a^2 + z^2)}\right). \tag{2.30}$$

The same conditions can be obtained by studying the cubic resolvent associated with the equation $\Delta = 0$:

$$u^3 - u\tilde{p} - \tilde{q} = 0, \tag{2.31}$$

where $\tilde{p} = 4w + \frac{p^2}{3}, \tilde{q} = \frac{2p^3}{27} + q^2 - \frac{8pw}{3}$, and where $p = -(l^2 - a^2), q = 2ml^2, w = -l^2(a^2 + z^2)$. The solutions of (2.31) are all real iff

$$\frac{\tilde{q}^2}{4} - \frac{\tilde{p}^3}{27} \leqslant 0, \tag{2.32}$$

i.e. one has to impose $\tilde{p} > 0$ (which amounts to (2.27)) and

$$\left[2m^2l^4 - \frac{4}{3}l^2(l^2 - a^2)(a^2 + z^2) - \frac{1}{27}(l^2 - a^2)^3\right]^2 - \frac{1}{27}\left[\frac{(l^2 - a^2)^2}{3} - 4l^2(a^2 + z^2)\right]^3 \leqslant 0. \tag{2.33}$$

The above inequality is implemented for $m^-_{\text{crit}} \leqslant m \leqslant m^+_{\text{crit}}$ and for $-m^+_{\text{crit}} \leqslant m \leqslant -m^-_{\text{crit}}$. The latter solution would correspond to negative values of the mass parameter m. Note also that for $m = m^+_{\text{crit}}$ one would obtain a black hole with $r_+ = r_c$. We do not discuss the latter case herein.

3. The Dirac equation

The Dirac equation for a charged massive particle of mass μ and electric charge e is

$$(\mathrm{i}\gamma^\mu D_\mu - \mu)\psi = 0, \tag{3.1}$$

where D is the Koszul connection on the bundle $S \otimes U(1)$, S being the spin bundle over the Kerr–Newman–dS manifold, that is

$$D_\mu = \partial_\mu + \tfrac{1}{4}\omega^{ij}_\mu \Gamma_i \Gamma_j + \mathrm{i}eA_\mu. \tag{3.2}$$

Here $\omega^{ij} = \omega^i_k \eta^{kj}$ are the spin connection 1-forms associated with a vierbein v^i, such that $\mathrm{d}s^2 = \eta_{ij} v^i \otimes v^j$, η being the usual Minkowski metric. γ_μ are the local Dirac matrices, related to the point-independent Minkowskian Dirac matrices Γ_i by the relations $\gamma_\mu = v^i_\mu \Gamma_i$.

Here we use the representation

$$\Gamma^0 = \begin{pmatrix} \mathbb{O} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}, \qquad \vec{\Gamma} = \begin{pmatrix} \mathbb{O} & -\vec{\sigma} \\ \vec{\sigma} & \mathbb{O} \end{pmatrix}, \tag{3.3}$$

where

$$\mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.4}$$

and $\vec{\sigma}$ are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

Thus

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2g_{\mu\nu}. \tag{3.6}$$

Following the general results of [14] one can obtain variable separation as in [1]. We limit ourselves to display the final result herein. The Petrov type D condition ensures the existence of a phase function $\mathcal{B}(r, \theta)$ such that

$$\mathrm{d}\mathcal{B} = \frac{1}{4Z(r, \theta)}\left(-2a\frac{\cos\theta}{\Xi}\,\mathrm{d}r - 2ar\frac{\sin\theta}{\Xi}\,\mathrm{d}\theta\right), \tag{3.7}$$

which indeed gives

$$\mathcal{B}(r,\theta) = \frac{\mathrm{i}}{4}\log\frac{r-\mathrm{i}a\cos\theta}{r+\mathrm{i}a\cos\theta}. \tag{3.8}$$

Now let us write the Dirac equation as

$$H_D\psi = 0. \tag{3.9}$$

Under a transformation $\psi \mapsto S^{-1}\psi$, with

$$S = Z^{-\frac{1}{4}}\,\mathrm{diag}(\mathrm{e}^{\mathrm{i}\mathcal{B}}, \mathrm{e}^{\mathrm{i}\mathcal{B}}, \mathrm{e}^{-\mathrm{i}\mathcal{B}}, \mathrm{e}^{-\mathrm{i}\mathcal{B}}), \tag{3.10}$$

it changes as

$$S^{-1}H_D S(S^{-1}\psi) = 0. \tag{3.11}$$

If we multiply this equation times

$$U = \mathrm{i}Z^{\frac{1}{2}}\,\mathrm{diag}(\mathrm{e}^{2\mathrm{i}\mathcal{B}}, -\mathrm{e}^{2\mathrm{i}\mathcal{B}}, -\mathrm{e}^{-2\mathrm{i}\mathcal{B}}, \mathrm{e}^{-2\mathrm{i}\mathcal{B}}), \tag{3.12}$$

and introduce the new wavefunction

$$\tilde{\psi} = (\Delta_\theta\Delta_r)^{\frac{1}{4}}S^{-1}\psi, \tag{3.13}$$

then the Dirac equation takes the form

$$(\mathcal{R}(r) + \mathcal{A}(\theta))\tilde{\psi} = 0, \tag{3.14}$$

where

$$\mathcal{R} = \begin{pmatrix} \mathrm{i}\mu r & 0 & -\sqrt{\Delta_r}\mathcal{D}_+ & 0 \\ 0 & -\mathrm{i}\mu r & 0 & -\sqrt{\Delta_r}\mathcal{D}_- \\ -\sqrt{\Delta_r}\mathcal{D}_- & 0 & -\mathrm{i}\mu r & 0 \\ 0 & -\sqrt{\Delta_r}\mathcal{D}_+ & 0 & \mathrm{i}\mu r \end{pmatrix}, \tag{3.15}$$

$$\mathcal{A} = \begin{pmatrix} -a\mu\cos\theta & 0 & 0 & -\mathrm{i}\sqrt{\Delta_\theta}\mathcal{L}_- \\ 0 & a\mu\cos\theta & -\mathrm{i}\sqrt{\Delta_\theta}\mathcal{L}_+ & 0 \\ 0 & -\mathrm{i}\sqrt{\Delta_\theta}\mathcal{L}_- & -a\mu\cos\theta & 0 \\ -\mathrm{i}\sqrt{\Delta_\theta}\mathcal{L}_+ & 0 & 0 & a\mu\cos\theta \end{pmatrix} \tag{3.16}$$

and

$$\mathcal{D}_\pm = \partial_r \pm \frac{1}{\Delta_r}((r^2+a^2)\partial_t - a\Xi\partial_\phi + \mathrm{i}eq_e r), \tag{3.17}$$

$$\mathcal{L}_\pm = \partial_\theta + \frac{1}{2}\cot\theta \pm \frac{\mathrm{i}}{\Delta_\theta\sin\theta}(\Xi\partial_\phi - a\sin^2\theta\,\partial_t + \mathrm{i}eq_m\cos\theta). \tag{3.18}$$

The separation of variables can then be obtained by searching for solutions of the form

$$\tilde{\psi}(t,\phi,r,\theta) = \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{-\mathrm{i}k\phi}\begin{pmatrix} R_1(r)S_2(\theta) \\ R_2(r)S_1(\theta) \\ R_2(r)S_2(\theta) \\ R_1(r)S_1(\theta) \end{pmatrix}, \qquad k \in \mathbb{Z} + \frac{1}{2}. \tag{3.19}$$

4. Hamiltonian formulation

The Hamiltonian for the Dirac equation can be read from (3.14) rewriting it in the form [2]

$$\mathrm{i}\partial_t\tilde{\psi} = H\tilde{\psi}. \tag{4.1}$$

Indeed we find

$$H = \left[\left(1 - \frac{\Delta_r}{\Delta_\theta}\frac{a^2\sin^2\theta}{(r^2+a^2)^2}\right)^{-1}\left(\mathbb{I}_4 - \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\theta}}\frac{a\sin\theta}{r^2+a^2}BC\right)\right](\tilde{\mathcal{R}}+\tilde{\mathcal{A}}), \tag{4.2}$$

where $\mathbb{I}_4$ is the 4×4 identity matrix,

$$\tilde{\mathcal{R}} = -\frac{\mu r\sqrt{\Delta_r}}{r^2+a^2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} + \begin{pmatrix}\mathcal{E}_- & 0 & 0 & 0\\ 0 & -\mathcal{E}_+ & 0 & 0\\ 0 & 0 & -\mathcal{E}_+ & 0\\ 0&0&0&\mathcal{E}_-\end{pmatrix}, \tag{4.3}$$

$$\tilde{\mathcal{A}} = \frac{a\mu\cos\theta\sqrt{\Delta_r}}{r^2+a^2}\begin{pmatrix}0&0&\mathrm{i}&0\\0&0&0&\mathrm{i}\\-\mathrm{i}&0&0&0\\0&-\mathrm{i}&0&0\end{pmatrix} + \begin{pmatrix}0 & -\mathcal{M}_- & 0 & 0\\ \mathcal{M}_+ & 0 & 0 & 0\\ 0 & 0 & 0 & \mathcal{M}_-\\ 0&0&-\mathcal{M}_+&0\end{pmatrix}, \tag{4.4}$$

where

$$\mathcal{E}_\pm = \mathrm{i}\frac{\Delta_r}{a^2+r^2}\left[\partial_r \mp \frac{a\Xi}{\Delta_r}\partial_\phi \pm \mathrm{i}\frac{eq_e r}{\Delta_r}\right], \tag{4.5}$$

$$\mathcal{M}_\pm = \frac{\sqrt{\Delta_r}\sqrt{\Delta_\theta}}{r^2+a^2}\left[\partial_\theta + \frac{1}{2}\cot\theta \pm \frac{\mathrm{i}\Xi}{\Delta_\theta\sin\theta}\partial_\phi \mp \frac{eq_m\cot\theta}{\Delta_\theta}\right], \tag{4.6}$$

and

$$B = \begin{pmatrix}0&0&-\mathrm{i}&0\\0&0&0&\mathrm{i}\\\mathrm{i}&0&0&0\\0&-\mathrm{i}&0&0\end{pmatrix}, \qquad C = \begin{pmatrix}0&0&0&\mathrm{i}\\0&0&-\mathrm{i}&0\\0&\mathrm{i}&0&0\\-\mathrm{i}&0&0&0\end{pmatrix} \tag{4.7}$$

satisfy $[B, C] = 0$, $B^2 = C^2 = \mathbb{I}_4$. Cf also [2] for the Kerr–Newman case. We need now to specify the Hilbert space. We do it as follows, in strict analogy with [1]. If we foliate spacetime in $t =$ constant slices $\mathcal{S}_t$, the metric on any slice (considering the shift vectors) is

$$\mathrm{d}\gamma^2 = \gamma_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta, \tag{4.8}$$

where $\alpha = 1, 2, 3$ and

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta}}{g_{00}}, \tag{4.9}$$

and local measure

$$\mathrm{d}\mu_3 = \sqrt{\det\gamma}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{\sin\theta}{\Xi}\frac{\rho^3}{\sqrt{\Delta_r - a^2\Delta_\theta\sin^2\theta}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{4.10}$$

In particular, the four-dimensional measure factors as

$$\mathrm{d}\mu_4 = \sqrt{-g_{00}}\,\mathrm{d}\mu_3\,\mathrm{d}t. \tag{4.11}$$

The action for a massless uncharged Dirac particle is then

$$S = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathcal{S}_t} \sqrt{-g_{00}}\,{}^t\psi^* \Gamma^0 \gamma^\mu D_\mu \psi \, \mathrm{d}\mu_3, \tag{4.12}$$

where the star indicates complex conjugation. Here by $\mathcal{S}_t$ we mean the range of coordinates parameterizing the region external to the event horizon: $r > r_+$, that is $\mathcal{S}_t := \mathcal{S} = (r_+, r_c) \times (0, \pi) \times (0, 2\pi)$. Then, the scalar product between wavefunctions should be

$$\langle \psi | \chi \rangle = \int_{\mathcal{S}} \sqrt{-g_{00}}\,{}^t\psi^* \Gamma^0 \gamma^t \chi \, \mathrm{d}\mu_3. \tag{4.13}$$

Let us also define

$$\alpha(r, \theta) := \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\theta}} \frac{a \sin\theta}{r^2 + a^2}. \tag{4.14}$$

We can now use (4.10), (3.13) and the relation

$$\gamma^2 = e_0^t \Gamma^0 + e_1^t \Gamma^1 \tag{4.15}$$

to express the product in the space of reduced wavefunctions (i.e. (3.13)):

$$\langle \tilde{\psi} | \tilde{\chi} \rangle = \int_{r_+}^{r_c} \mathrm{d}r \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, \frac{r^2 + a^2}{\Delta_r} \frac{\sin\theta}{\sqrt{\Delta_\theta}}\,{}^t\tilde{\psi}^* (\mathbb{I}_4 + \alpha(r, \theta) BC) \tilde{\chi}, \tag{4.16}$$

where a factor $\Xi^{-\frac{1}{2}}$ has been dropped. The matrix in the parenthesis in the previous equation is the inverse of the one in the square brackets in (4.2), and it represents an improvement to the dS case of the matrix which has been introduced in [2] for the Kerr–Newman case.

The above scalar product is positive definite, as we show in the following. Being ± 1 the eigenvalues of BC, we need to prove that

$$\eta := \sup_{r \in (r_+, r_c), \theta \in (0, \pi)} \alpha(r, \theta) < 1. \tag{4.17}$$

We can write $\alpha(r, \theta) = \beta(r)\gamma(\theta)$, with

$$\gamma(\theta) = \frac{\sin\theta}{\sqrt{\Delta_\theta}}. \tag{4.18}$$

Then

$$\gamma'(\theta) = \frac{\cos\theta}{\Delta_\theta^{\frac{3}{2}}} \left(1 + \frac{a^2}{l^2}\right), \tag{4.19}$$

so that γ reaches its maximum at $\theta = \pi/2$ and

$$\gamma(\pi/2) = 1, \tag{4.20}$$

which implies

$$\alpha(r, \theta) \leqslant \beta(r). \tag{4.21}$$

Next, from

$$0 = \Delta_r(r_+) \tag{4.22}$$

we have

$$z^2 - 2mr_+ = -\left(r_+^2 + a^2\right)(l^2 - r^2)/l^2 < 0, \tag{4.23}$$

and then, for $r_+ \leqslant r \leqslant r_c$ we have $z^2 - 2mr < 0$ (note that $l^2 > r_c^2$ for our case). Thus

$$\beta^2(r) = \frac{a^2 \Delta_r}{(r^2 + a^2)^2} = \frac{a^2}{l^2} \frac{l^2 - r^2}{r^2 + a^2} + a^2 \frac{z^2 - 2mr}{(r^2 + a^2)^2} \leqslant \frac{a^2}{l^2} \frac{l^2 - r^2}{r^2 + a^2} =: h(r). \tag{4.24}$$

Now, the last function is a decreasing function of r, so that for $r \geqslant r_+ > 0$ we have $h(r_c) \leqslant h(r) \leqslant h(r_+) < h(0)$, so that

$$\beta^2(r) \leqslant h(r_+) = \frac{a^2}{l^2}\frac{l^2 - r_+^2}{r_+^2 + a^2} < h(0) = 1, \tag{4.25}$$

and then

$$\eta \leqslant \sqrt{h(r_+)} < 1. \tag{4.26}$$

5. Essential self-adjointness of $\hat{H}$

We follow strictly our analysis in [1], limiting ourselves to some essential definitions and results. Let us introduce the space of functions $\mathcal{L}^2 := (L^2((r_+, r_c) \times S^2; \mathrm{d}\mu))^4$ with measure

$$\mathrm{d}\mu = \frac{r^2 + a^2}{\Delta_r}\frac{\sin\theta}{\sqrt{\Delta_\theta}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{5.1}$$

and define $\mathcal{H}_{\langle\rangle}$ as the Hilbert space $\mathcal{L}^2$ with the scalar product (4.16). We will also consider a second Hilbert space $\mathcal{H}_{()}$, which is obtained from $\mathcal{L}^2$ with the scalar product

$$(\psi|\chi) = \int_{r_+}^{r_c}\mathrm{d}r\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,\frac{r^2 + a^2}{\Delta_r}\frac{\sin\theta}{\sqrt{\Delta_\theta}}{}^t\,\psi^*\chi = \int \mathrm{d}\mu^t\,\psi^*\chi. \tag{5.2}$$

It is straightforward to show that $\|\cdot\|_{\langle\rangle}$ and $\|\cdot\|_{()}$ are equivalent norms. It is also useful to introduce $\hat{\Omega}^2 : \mathcal{L}^2 \to \mathcal{L}^2$ as the multiplication operator by $\Omega^2(r,\theta)$:

$$\Omega^2(r,\theta) := \mathbb{I}_4 + \alpha(r,\theta)BC. \tag{5.3}$$

Then we have

$$\langle\psi|\chi\rangle = \int \mathrm{d}\mu^t\,\psi^*\Omega^2\chi = (\psi|\hat{\Omega}^2\chi). \tag{5.4}$$

We introduce also $\hat{\Omega}^{-2} : \mathcal{L}^2 \to \mathcal{L}^2$ as the multiplication operator by Ω^{-2}

$$\Omega^{-2}(r,\theta) := \frac{1}{1 - \alpha^2(r,\theta)}(\mathbb{I}_4 - \alpha(r,\theta)BC), \tag{5.5}$$

and analogously $\hat{\Omega}$, $\hat{\Omega}^{-1}$ are defined as operators from $\mathcal{L}^2$ to $\mathcal{L}^2$ which multiply by $\Omega(r,\theta)$, $\Omega^{-1}(r,\theta)$ respectively, where Ω and Ω^{-1} are defined as the principal square root of Ω^2 and Ω^{-2} respectively. They are injective and surjective. As operators from $\mathcal{H}_{()}$ to $\mathcal{H}_{()}$, $\hat{\Omega}^2$, $\hat{\Omega}^{-2}$, $\hat{\Omega}$, $\hat{\Omega}^{-1}$ are bounded, positive and self-adjoint.

Let us set $H_0 := \tilde{\mathcal{R}} + \tilde{\mathcal{A}}$, which is formally self-adjoint on $\mathcal{H}_{()}$, and define the operator $\hat{H}_0$ on $\mathcal{L}^2$ with

$$\begin{aligned} &D(\hat{H}_0) = C_0^\infty((r_+, r_c) \times S^2)^4 =: \mathcal{D}, \\ &\hat{H}_0\chi = H_0\chi, \qquad \chi \in \mathcal{D}. \end{aligned} \tag{5.6}$$

Note that $\mathcal{D}$ is dense in $\mathcal{H}_{()}$. Let us point out that for the formal differential expression H in (4.2), which is formally self-adjoint on $\mathcal{H}_{\langle\rangle}$, one can write $H = \Omega^{-2}H_0$. Then we define on $\mathcal{L}^2$ the differential operator $\hat{H} = \hat{\Omega}^{-2}\hat{H}_0$, with

$$\begin{aligned} &D(\hat{H}) = \mathcal{D}, \\ &\hat{H}\chi = H\chi, \qquad \chi \in \mathcal{D}. \end{aligned} \tag{5.7}$$

The same considerations as in [1] lead to the following conclusions: $\hat{H}$ is essentially self-adjoint if and only if $\hat{H}_0$ is essentially self-adjoint on the same domain (in different Hilbert spaces); cf theorem 1 in [1]. See also [15] for the Kerr–Newman case. Moreover, one can show by means of variable separation (cf [1]) that $\hat{H}_0$ is essentially self-adjoint. As a consequence, there exists a unique self-adjoint extension $\hat{T}_{H_0}$ with domain $\mathfrak{D} \subset \mathcal{H}_{()}$ and, correspondingly, a unique self-adjoint extension $\hat{T}_H := \hat{\Omega}^{-2}\hat{T}_{H_0}$ of $\hat{H}$ on $\mathfrak{D} \subset \mathcal{H}_{\langle\rangle}$ (note that it is the same domain on two different Hilbert spaces). We do not report the details of the variable separation process, because they are the same as in [1]. We limit ourselves to sketch the main points. By means of the unitary transformation

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 & \mathrm{i} \\ \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \tag{5.8}$$

we get

$$V H_0 V^* = \begin{pmatrix} \frac{1}{r^2+a^2}(\mathrm{i}a\,\Xi\partial_\phi + \mu_+(r))\mathbb{I} & \frac{\Delta_r}{r^2+a^2}\partial_r\mathbb{I} + \frac{\sqrt{\Delta_r}}{r^2+a^2}\mathbb{U} \\ -\frac{\Delta_r}{r^2+a^2}\partial_r\mathbb{I} + \frac{\sqrt{\Delta_r}}{r^2+a^2}\mathbb{U} & \frac{1}{r^2+a^2}(\mathrm{i}a\,\Xi\partial_\phi + \mu_-(r))\mathbb{I} \end{pmatrix}, \tag{5.9}$$

where

$$\mu_\pm(r) := eq_e r \pm \mu r\sqrt{\Delta_r}, \tag{5.10}$$

and where $\mathbb{U}$ is the 2×2 matrix formal differential expression

$$\mathbb{U} = \begin{pmatrix} -\mu a\cos(\theta) & \mathrm{i}\sqrt{\Delta_\theta}\left(\partial_\theta + \frac{1}{2}\cot(\theta) + g\right) \\ \mathrm{i}\sqrt{\Delta_\theta}\left(\partial_\theta + \frac{1}{2}\cot(\theta) - g\right) & \mu a\cos(\theta) \end{pmatrix}, \tag{5.11}$$

with $g := i\,\frac{1}{\Delta_\theta \sin(\theta)}\,\Xi\partial_\phi - \frac{1}{\Delta_\theta}q_m e\cot(\theta)$.

Then the following variable separation ansatz:

$$V\chi(r,\theta,\phi) = \frac{\mathrm{e}^{-\mathrm{i}k\phi}}{\sqrt{2\pi}} V \begin{pmatrix} R_1(r)S_2(\theta) \\ R_2(r)S_1(\theta) \\ R_2(r)S_2(\theta) \\ R_1(r)S_1(\theta) \end{pmatrix}, \tag{5.12}$$

with $k \in \mathbb{Z} + \frac{1}{2}$, leads to the following reduction of the angular part: by defining $b_k(\theta) := \frac{1}{\Delta_\theta \sin(\theta)}\,\Xi k - \frac{1}{\Delta_\theta}q_m e\cot(\theta)$, one finds that the operator $\hat{\mathbb{U}}_k$ defined on $D(\hat{\mathbb{U}}_k) = C_0^\infty(0,\pi)^2$ and whose formal differential expression is

$$\mathbb{U}_k = \begin{pmatrix} -\mu a\cos(\theta) & \mathrm{i}\sqrt{\Delta_\theta}\left(\partial_\theta + \frac{1}{2}\cot(\theta) + b_k(\theta)\right) \\ \mathrm{i}\sqrt{\Delta_\theta}\left(\partial_\theta + \frac{1}{2}\cot(\theta) - b_k(\theta)\right) & \mu a\cos(\theta) \end{pmatrix},$$

is essentially self-adjoint for any $k \in \mathbb{Z} + \frac{1}{2}$ for $\frac{q_m e}{\Xi} \in \mathbb{Z}$. If one considers the self-adjoint extension $\bar{\hat{\mathbb{U}}}_k$ of $\hat{\mathbb{U}}_k$, one can show that $\bar{\hat{\mathbb{U}}}_k$ has purely discrete spectrum which is simple (see [1]).

Let us introduce the (normalized) eigenfunctions $S_{k;j}(\theta) := \binom{S_{1k;j}(\theta)}{S_{2k;j}(\theta)}$ of the operator $\bar{\hat{\mathbb{U}}}_k$:

$$\bar{\hat{\mathbb{U}}}_k \begin{pmatrix} S_{1k;j}(\theta) \\ S_{2k;j}(\theta) \end{pmatrix} = \lambda_{k;j} \begin{pmatrix} S_{1k;j}(\theta) \\ S_{2k;j}(\theta) \end{pmatrix}, \tag{5.13}$$

then $\mathcal{H}_{k,j} := L^2\big((r_+, r_c), \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r\big)^2 \otimes M_{k,j}$, where $M_{k,j} := \{F_{k;j}(\theta,\phi)\}$, with $F_{k;j}(\theta,\phi) := S_{k;j}(\theta)\frac{\mathrm{e}^{-\mathrm{i}k\phi}}{\sqrt{2\pi}}$, is such that the eigenvalue equation for $V\hat{H}_0V^*$ becomes equivalent to the following 2×2 Dirac system for the radial part (cf [1]):

$$\begin{pmatrix} \frac{1}{r^2+a^2}(a\Xi k + \mu_+(r)) & \frac{\Delta_r}{r^2+a^2}\partial_r + \frac{\sqrt{\Delta_r}}{r^2+a^2}\lambda_{k;j} \\ -\frac{\Delta_r}{r^2+a^2}\partial_r + \frac{\sqrt{\Delta_r}}{r^2+a^2}\lambda_{k;j} & \frac{1}{r^2+a^2}(a\Xi k + \mu_-(r)) \end{pmatrix} X = \omega X, \tag{5.14}$$

where $X(r) := \binom{X_1(r)}{X_2(r)}$, and we introduce a radial Hamiltonian $\hat{h}_{k,j}$, which is defined on $\mathcal{D}_{k,j} := C_0^\infty(r_+, r_c)^2$ and has the following formal expression:

$$h_{k,j} := \begin{pmatrix} \frac{1}{r^2+a^2}(a\Xi k + \mu_+(r)) & \frac{\Delta_r}{r^2+a^2}\partial_r + \frac{\sqrt{\Delta_r}}{r^2+a^2}\lambda_{k;j} \\ -\frac{\Delta_r}{r^2+a^2}\partial_r + \frac{\sqrt{\Delta_r}}{r^2+a^2}\lambda_{k;j} & \frac{1}{r^2+a^2}(a\Xi k + \mu_-(r)) \end{pmatrix}. \tag{5.15}$$

In the following, we study essential self-adjointness conditions for the reduced Hamiltonian $\hat{h}_{k,j}$.

Essential self-adjointness of $\hat{h}_{k,j}$. The differential expression $h_{k,j}$ is formally self-adjoint in the Hilbert space $L^2\big((r_+, r_c), \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r\big)^2$. In order to study the essential self-adjointness of the reduced Hamiltonian in $C_0^\infty(r_+, r_c)^2$ one has to check if the limit point case occurs both at the event horizon $r = r_+$ and at $r = r_c$. We show that the following result holds.

Theorem 1. *$\hat{h}_{k,j}$ is essentially self-adjoint on $C_0^\infty(r_+, r_c)^2$.*

Proof. We choose the tortoise coordinate y defined by

$$\mathrm{d}y = \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r, \tag{5.16}$$

and obtain $y \in \mathbb{R}$ with $y \to \infty$ as $r \to r_c$ and $y \to -\infty$ as $r \to r_+$. Then we get

$$h_{k,j} = \begin{pmatrix} 0 & \partial_y \\ -\partial_y & 0 \end{pmatrix} + V(r(y)), \tag{5.17}$$

and the corollary to theorem 6.8, p 99 in [16] ensures that the limit point case holds for $h_{k,j}$ at $y = \infty$. The same corollary can be used also for concluding that the limit point case occurs also for $y = -\infty$ and this allows us to claim that the above theorem holds true. □

It is also useful to point out that it holds

$$\lim_{y\to-\infty} V(r(y)) = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_+ \end{pmatrix}, \tag{5.18}$$

where

$$\varphi_+ := \frac{1}{r_+^2+a^2}(ak\Xi + eq_e r_+), \tag{5.19}$$

and that

$$\lim_{y\to\infty} V(r(y)) = \begin{pmatrix} \varphi_c & 0 \\ 0 & \varphi_c \end{pmatrix}, \tag{5.20}$$

where

$$\varphi_c := \frac{1}{r_c^2+a^2}(ak\Xi + eq_e r_c). \tag{5.21}$$

6. The non-existence of time-periodic normalizable solutions

As it is well-known from the study of the Kerr–Newman case and of the Kerr–Newman–AdS case [1, 5, 6, 8, 17], eigenvalues for the Hamiltonian H correspond to the solutions of the following system of coupled eigenvalue equations have to be satisfied simultaneously in $L^2\big((0,\pi), \frac{\sin(\theta)}{\sqrt{\Delta_\theta}}\,\mathrm{d}\theta\big)^2$ and in $L^2\big((r_+, r_c), \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r\big)^2$ respectively:

$$\hat{\bar{\mathbb{U}}}_{k\omega} S = \lambda S \tag{6.1}$$

and

$$\hat{\bar{h}}_{k,j} X = \omega X. \tag{6.2}$$

Note that the Dirac equation (3.14) in the Chandrasekhar-like variable separation ansatz (3.19) reduces to the couple of equations (6.1) and (6.2).

The spectrum of the angular momentum operator $\hat{\bar{\mathbb{U}}}_{k\omega}$ is discrete for any $\omega \in \mathbb{R}$, as it can be shown in a step-by-step replication of the calculations appearing in [1]. We show that both in the non-extremal case and in the extremal one the radial Hamiltonian $\hat{\bar{h}}_{k,j}$ for any $\lambda_{k;j}$ has a spectrum that is absolutely continuous and coincides with $\mathbb{R}$, and then we infer that no eigenvalue of $\hat{\bar{H}}$ exists. As a consequence (cf remark 1 in [1]), we can exclude the existence of normalizable time-periodic solutions of the Dirac equation.

Spectrum of the operator $\hat{\bar{h}}_{k,j}$. In order to study the spectral properties of $\hat{\bar{h}}_{k,j}$, as in [1] we introduce two auxiliary self-adjoint operators $\hat{h}_{\mathrm{hor}}$ and $\hat{h}_{r_c}$:

$$\begin{aligned}
&D(\hat{h}_{\mathrm{hor}}) = \big\{X \in L^2_{(r_+,r_0)},\ X \text{ is locally absolutely continuous; } B(X)=0;\ \hat{h}_{\mathrm{hor}}X \in L^2_{(r_+,r_0)}\big\},\\
&\hat{h}_{\mathrm{hor}} X = h_{k,j} X;\\
&D(\hat{h}_{r_c}) = \big\{X \in L^2_{(r_0,r_c)},\ X \text{ is locally absolutely continuous; } B(X)=0;\ \hat{h}_{r_c}X \in L^2_{(r_0,r_c)}\big\}\\
&\hat{h}_{r_c} X = h_{k,j} X.
\end{aligned} \tag{6.3}$$

r_0 is an arbitrary point with $r_+ < r_0 < r_c$, at which the boundary condition $B(X) := X_1(r_0) = 0$ is imposed. We also have defined $L^2_{(r_+,r_0)} := L^2\big((r_+, r_0), \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r\big)^2$ and $L^2_{(r_0,r_c)} := L^2\big((r_0, r_c), \frac{r^2+a^2}{\Delta_r}\,\mathrm{d}r\big)^2$. Note that we omit the indices k, j for these operators.

As to the spectral properties of $\hat{h}_{r_c}$, a suitable change of coordinates consists in introducing a tortoise-like coordinate defined by equation (5.16). It is then easy to show that the following result holds.

Lemma 1. $\sigma_{ac}(\hat{h}_{r_c}) = \mathbb{R}$.

Proof. The proof is completely analogous to that of lemma 3 in [1]. We still provide the details. Theorem 16.7 of [16] allows us to find that the spectrum of $\hat{h}_{r_c}$ is absolutely continuous in $\mathbb{R} - \{\varphi_c\}$. This can be proved as follows. Let us write the potential $V(r(y))$ in (5.17)

$$V(r(y)) = \begin{pmatrix} \varphi_c & 0 \\ 0 & \varphi_c \end{pmatrix} + P_2(r(y)), \tag{6.4}$$

which implicitly defines $P_2(r(y))$. The first term on the left-hand side of (6.4) is of course of bounded variation; on the other hand, $|P_2(r(y))| \in L^1(d,\infty)$, with $d \in (y(r_0),\infty)$. As a consequence, the hypotheses of theorem 16.7 in [16] are trivially satisfied, and one finds that the spectrum of $\hat{h}_{\mathrm{hor}}$ is absolutely continuous in $\mathbb{R} - \{\varphi_c\}$.

We show also that φ_c is not an eigenvalue of $\hat{h}_{r_c}$. As in the Kerr–Newman case (cf [5]), one needs simply to study the asymptotic behavior of the solutions of the linear system

$$X' = \begin{pmatrix} -\lambda_{k;j}\dfrac{\sqrt{\Delta_r}}{r^2+a^2} & \varphi_c - \dfrac{1}{r^2+a^2}(a\Xi k + \mu_-(r)) \\ \dfrac{1}{r^2+a^2}(a\Xi k + \mu_+(r)) - \varphi_c & \lambda_{k;j}\dfrac{\sqrt{\Delta_r}}{r^2+a^2} \end{pmatrix} X$$
$$=: \bar{R}(r(y))X, \tag{6.5}$$

where $r = r(y)$ and where the prime indicates the derivative with respect to y. One easily realizes that

$$\int_{\mathrm{d}}^{\infty} \mathrm{d}y |\bar{R}(r(y))| < \infty, \tag{6.6}$$

and then (cf Levinson theorem e.g. in [18]: theorem 1.3.1, p 8) one can find two linearly independent asymptotic solutions as $y \to \infty$ whose leading order is given by $X_I = \binom{1}{0}$ and $X_{II} = \binom{0}{1}$. As a consequence no normalizable solution of equation (6.5) can exists, and then φ_c cannot be an eigenvalue. □

The following result holds.

Theorem 2. $\sigma_{ac}(\bar{\hat{h}}_{k,j}) = \mathbb{R}$.

Proof. Thanks to standard decomposition methods for the absolutely continuous spectrum (see the appendix) the proof is trivial, because the absolutely continuous part of $\bar{\hat{h}}_{k,j}$ is unitarily equivalent to the absolutely continuous part of $\hat{h}_{\mathrm{hor}} \oplus \hat{h}_{r_c}$. As a consequence, $\sigma_{ac}(\bar{\hat{h}}_{k,j}) = \sigma_{ac}(\hat{h}_{\mathrm{hor}}) \cup \sigma_{ac}\left(\hat{h}_{r_c}\right)$. The latter set is $\mathbb{R}$ (see lemma 1). □

The presence of a cosmological horizon which is non-degenerate (i.e. it corresponds to a simple zero of Δ_r) is as seen the main ingredient for the above conclusion. A rough explanation for this result is that such a presence forbids the possibility to get normalizability of the time-periodic solutions. The rationale beyond it is the above result concerning the absolutely continuous spectrum.

7. Conclusions

By extending the results obtained in [1], we have shown that the Dirac Hamiltonian for a charged particle in the background of a Kerr–Newman–de Sitter black hole is essentially self-adjoint on $C_0^\infty((r_+, r_c) \times S^2)^4$. Moreover, our analysis has revealed that the point spectrum of the Hamiltonian is empty, which is equivalent to the condition for the absence of normalizable time-periodic solutions of the Dirac equation. The latter result has been shown to hold true even in the extremal case, which is usually much harder to be checked, in a rather straightforward way, and the role of the (non-degenerate) cosmological event horizon in this respect has been pointed out.

Appendix. Decomposition method for the absolutely continuous spectrum

For the sake of completeness, we give some more details about the decomposition method (or the splitting method) [16, 19–21] as applied in the analysis of the absolutely continuous spectrum. It is surely known to experts, but perhaps not so explicitly written in the literature.

The following proof is essentially an extended version, trivially adapted to the Dirac case, of the proof appearing at p 239 of [16] for the Sturm–Liouville case, and it also appeals to the proof of Korollar 6.2 in [22].

Let us consider a Dirac system with formal differential expression τ, formally self-adjoint in a suitable Hilbert space which we indicate with $L_2(a,b)$ for short, with $(a,b) \subset \mathbb{R}$. See [16, 22] for more details. Let us introduce the maximal operator $\hat{K}$ associated with the formal expression τ, with domain

$$D(\hat{K}) = \{X \in L_2(a,b),\ X \text{ is locally absolutely continuous; } \hat{K}X \in L_2(a,b)\},$$

and the minimal operator $\hat{K}_0$ defined as the closure of the operator $\hat{K}_0'$ defined on

$$D(\hat{K}_0') = \{X \in D(\hat{K});\ X \text{ has compact support in } (a,b)\},$$

and with formal expression τ. For an explicit characterization of $\hat{K}_0$ see also [16]. Let $\hat{T}$ be a self-adjoint extension of $\hat{K}_0$.

Let us also define (cf Korollar 6.2 in [22]) $\hat{K}_{00}$ as the operator with the same formal expression τ and domain $D(\hat{K}_{00}) = \{X \in D(\hat{K}_0);\ X(c) = 0\}$, with $c \in (a,b)$. If $\hat{K}_{a,0}, \hat{K}_{b,0}$ are the minimal operators associated with τ in $L_2(a,c)$ and $L_2(c,b)$ respectively, one has $\hat{K}_{00} = \hat{K}_{a,0} \oplus \hat{K}_{b,0}$. Let $\hat{T}_a$ and $\hat{T}_b$ be self-adjoint extensions of $\hat{K}_{a,0}, \hat{K}_{b,0}$; then both $\hat{T}$ and $\hat{T}_a \oplus \hat{T}_b$ are finite-dimensional extensions of $\hat{K}_{00}$. (Incidentally, this is enough for concluding that the essential spectrum of $\hat{T}$ coincides with the essential spectrum of $\hat{T}_a \oplus \hat{T}_b$, which is part of the content of Korollar 6.2 in [22], and proves the splitting method for the essential spectrum). As a consequence, the difference of their resolvents $\hat{D} := (\hat{T} - \zeta I)^{-1} - (\hat{T}_a \oplus \hat{T}_b - \zeta I)^{-1}$, (with $\zeta \in \rho(\hat{T}) \cap \rho(\hat{T}_a \oplus \hat{T}_b)$), is an operator of finite rank (see [23], lemma 2, p 214). Then, according to the Kuroda–Birman theorem (see e.g. theorem XI.9, p 27 in [24]; see also [25]), the wave operators $\Omega^{\pm}(\hat{T}, \hat{T}_a \oplus \hat{T}_b)$ exist and are complete. As a consequence, the absolutely continuous part of $\hat{T}_a \oplus \hat{T}_b$ is unitarily equivalent to the absolutely continuous part of $\hat{T}$, and this in turn implies that $\sigma_{ac}(\hat{T}) = \sigma_{ac}(\hat{T}_a \oplus \hat{T}_b) = \sigma_{ac}(\hat{T}_a) \cup \sigma_{ac}(\hat{T}_b)$.

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